

# On the $p$ -median polytope of $Y$ -free graphs

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Received 21 June 2005; accepted 5 September 2006

Available online 29 October 2007

## Abstract

In this paper we consider a well-known class of valid inequalities for the  $p$ -median and the uncapacitated facility location polytopes, the odd cycle inequalities. It is known that their separation problem is polynomially solvable. We give a new polynomial separation algorithm based on a reduction from the original graph. Then, we define a non-trivial class of graphs, where the odd cycle inequalities together with the linear relaxations of both the  $p$ -median and uncapacitated facility location problems, suffice to describe the associated polytope. To do this, we first give a complete description of the fractional extreme points of the linear relaxation for the  $p$ -median polytope in this class of graphs.

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**Keywords:**  $p$ -median problem; Facility location; Polyhedral combinatorics

## 1. Introduction

Let  $G = (V, E)$  be a directed graph, not necessarily connected, where each arc  $(u, v) \in E$  has an associated cost  $c(u, v)$ . The  $p$ -median problem ( $p$ MP) consists of selecting  $p$  nodes, usually called *centers*, and then assigning each non-selected node to a selected node. The goal is to select  $p$  nodes that minimize the sum of the costs yielded by the assignment of the non-selected nodes. This problem has several applications such as location of bank accounts [9], placement of web proxies in a computer network [22], semistructured data bases [21,18]. When the number of centers is not specified and each opened center induces a given cost, this is called the *uncapacitated facility location problem* (UFLP).

In this paper we study the so-called *odd cycle inequalities*. We give a new separation algorithm, and we show that when they are added to the linear programming relaxation of the  $p$ MP, a complete description of the polytope is obtained for the so-called  $Y$ -free graphs. To accomplish this, first we have to characterize the extreme points of this linear programming relaxation for the class of  $Y$ -free graphs. Finally and for the same class of graphs, we show that when we add the odd cycle inequalities to the linear programming relaxation of the UFLP, we also obtain an integral polytope. We do not know of any other class of graphs for which the  $p$ -median polytope has been characterized.

The  $p$ MP is NP-hard in general [16]. For some particular cases it has been shown that it is polynomially solvable. This is the case when the underlying graph is an undirected tree and the cost function  $c$  is defined by  $c(u, v) = w(u)d(u, v)$ , where  $w(u)$  is a positive weight associated with each node  $u \in V$ , and  $d(u, v)$  is the length

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of the unique path in the tree from  $u$  to  $v$ , and it satisfies the triangle inequalities. An  $O(p|V|^2)$  algorithm has been presented in [20] improving a previous algorithm in the same class of graphs,  $O(p^2|V|^2)$  in [16] and  $O(p|V|^3)$  in [14]. When the tree is directed, an  $O(p^2L)$  algorithm based on dynamic programming is given in [22], where  $L$  is the path length of the tree.

Many heuristics without guarantee on the value of the solution have been given (see [9] for references). In [9] some known heuristics were evaluated using Lagrangian relaxation. This study has been used in [13] for evaluating heuristics based on a reduction from the set-covering problem. Later,  $\alpha$ -approximation algorithms were developed, where  $\alpha$  is not a constant factor, see [4–6]. The first constant-factor approximation algorithm was given in [8], with  $\alpha = 6\frac{2}{3}$ . Algorithms that achieve a performance within a ratio of 6 and 4 were presented in [15] and [7], respectively. Most of approximation algorithms are based on rounding the optimal fractional solution of the following natural linear programming relaxation for the  $p$ MP:

$$\text{minimize } \sum_{(u,v) \in E} c(u,v)x(u,v), \quad (1)$$

$$\sum_{v:(u,v) \in E} x(u,v) = 1 - y(u) \quad \forall u \in V, \quad (2)$$

$$\sum_{v \in V} y(v) = p, \quad (3)$$

$$x(u,v) \leq y(v) \quad \forall (u,v) \in E, \quad (4)$$

$$0 \leq y(v) \leq 1 \quad \forall v \in V, \quad (5)$$

$$x(u,v) \geq 0 \quad \forall (u,v) \in E. \quad (6)$$

If in addition the variables are 0-1, then we obtain an integer linear programming formulation. The 0-1 variable  $y(v)$ ,  $v \in V$  is 1 if the node  $v$  is selected as a center and 0 otherwise. The 0-1 variable  $x(u,v)$  takes the value 1 if a non-selected node  $u$  is assigned to a selected node  $v$ . Constraints (2) ensure that each node must be assigned to some center, constraints (3) ensure that exactly  $p$  centers must be selected and constraints (4) indicate that if a node  $v$  is not selected as center then no node  $u$  may be assigned to  $v$ .

Denote by  $P_p(G)$  the polytope defined by constraints (2)–(6), and let  $pMP(G)$  be the convex hull of  $P_p(G) \cap \{0, 1\}^{|E|+|V|}$ , this is called the  $p$ -median polytope of  $G$ .

Let  $Q(G)$  be the polytope defined by constraints (2), (4), (5) and (6). Let  $UFLP(G)$  be the convex hull of  $Q(G) \cap \{0, 1\}^{|E|+|V|}$ , this is the *uncapacitated facility location polytope* of  $G$ .

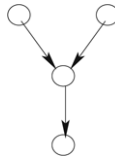
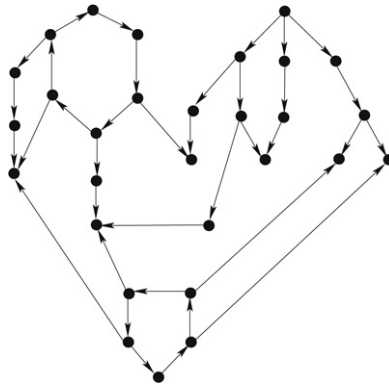
Some polyhedral properties of  $P_p(G)$  and the integrality gap are discussed in [23], when  $G$  is restricted to be a tree. This relaxation has also been studied for trees in [11]. In [12], an extended formulation by adding  $O(|V|^{p-1})$  variables characterizes the  $p$ -median polytope for any fixed  $p$ ; also the 2-median polytope is characterized in the original set of variables.

A formulation based on the arc variables is studied in [2]. Then the relationship with the stable set problem is exploited. This approach was also used in [10] for the UFLP. Also in [2], they remark that the  $p$ -median polytope when  $p = 2$  is completely described using a result in [17]. In [3] a Branch-and-Cut-and-Prize algorithm is developed to solve large instances for the  $p$ -median problem.

We conclude this introduction with a few definitions. For a vector  $x \in \mathbb{R}^S$  and a subset  $A \subseteq S$ , we denote  $\sum_{a \in A} x(a)$  by  $x(A)$ . For a set  $W \subset V$ , we denote by  $\delta^+(W)$ , the set of arcs  $(u,v) \in E$ , with  $u \in W$  and  $v \in V \setminus W$ , and by  $\delta^-(W)$  the set of arcs  $(u,v)$ , with  $v \in W$  and  $u \in V \setminus W$ . We write  $\delta^+(v)$  and  $\delta^-(v)$  instead of  $\delta^+(\{v\})$  and  $\delta^-(\{v\})$ , respectively. For a solution  $(x,y)$  of  $P_p(G)$ , define, for simplicity,  $z = (x,y)$  as  $z(u,v) = x(u,v)$  for  $(u,v) \in E$  and  $z(v) = y(v)$  for  $v \in V$ . If an inequality (4) is satisfied as equation by  $z$  with respect to an arc  $(u,v)$ , we say that  $(u,v)$  is *saturated* by  $z$ , otherwise it is *non-saturated* by  $z$ .

A directed graph  $G = (V, E)$ , not necessarily connected, is called a  $Y$ -free graph if  $(u,v) \in E$  implies  $(v,u) \notin E$  and it does not contain as a subgraph the graph of Fig. 1. This class of graphs may contain cycles and it contains the class of 1-rooted directed trees, Fig. 2 shows a  $Y$ -free graph.

This paper is organized as follows. In Section 2 we give a separation algorithm for the odd cycle inequalities. In Section 3 a complete description of the fractional extreme points of  $P_p(G)$  is given for  $Y$ -free graphs. In Section 4 we show that when we add the odd cycle inequalities to (2)–(6), we obtain a complete description of  $pMP(G)$  when

Fig. 1. The graph  $Y$ .Fig. 2. A  $Y$ -free graph.

$G$  is  $Y$ -free. In Section 5 it is proved that when Eq. (3) is removed from the description of  $pMP(G)$  and  $G$  is  $Y$ -free, then the polytope is still integral.

## 2. Odd cycle inequalities

In this section we describe this class of inequalities and their separation algorithm. Let  $C$  be the cycle

$$C = \{(v_i, v_{i+1}) | i = 1, \dots, 2l\} \cup \{(v_{2l+1}, v_1)\}.$$

The inequality

$$x(C) \leq \frac{|C| - 1}{2} \quad (7)$$

is called an *odd cycle inequality*.

**Lemma 1.** *Inequality (7) is valid for  $pMP(G)$  and for  $UFLP(G)$ .*

**Proof.** The combination of inequalities (2) and (4) gives

$$\begin{aligned} x(u_i, u_{i+1}) + x(\delta^+(u_{i+1})) &\leq 1, \quad \text{for } i = 1, \dots, 2l, \\ x(u_{2l+1}, u_1) + x(\delta^+(u_1)) &\leq 1. \end{aligned}$$

Adding these inequalities and non-negativity constraints we obtain

$$2x(C) \leq 2l + 1 = |C|.$$

Dividing by 2 and rounding off the right-hand side yield inequality (7).  $\square$

Odd cycle inequalities are a special case of  $Wq$  inequalities introduced in [3] with  $q = 1$ . Since we might have an exponential number of odd cycle inequalities, it is important to have an efficient algorithm to solve the *separation problem*: Given a vector  $\bar{x}$  satisfying (2)–(6), find a violated odd cycle inequality if there is any, or prove that none exists. We describe such a procedure below.

The odd cycle inequalities we consider here, are equivalent to the odd cycle inequalities that are valid for the stable set polytope in the *intersection graph*. The intersection graph has a node for every arc in  $E$ , and for any two arcs

of the form  $(u, v)$ ,  $(v, w)$  there is an edge in the intersection graph. Their separation can be reduced to  $|E|$  shortest path problems in an auxiliary graph with positive arc-weights obtained from the intersection graph, see Theorem 68.1 in [19] for instance. Below, we give a reduction to  $|V|$  shortest path problem in an auxiliary graph with no negative cycle, obtained from the original graph.

Inequality (7) can be written as

$$|C| - 2x(C) \geq 1. \quad (8)$$

To find a violated inequality, if there is any, we create an auxiliary graph as follows. For each node  $u$ , we make two copies  $u'$ ,  $u''$ . For each arc  $(u, v)$  we create arcs  $(u', v')$  and  $(u'', v')$  with weights  $1 - 2\bar{x}(u, v)$ . Then for each node  $u$  we find a shortest path  $P$  from  $u'$  to  $u''$ . We identify every node in  $P$  with its copy, this gives a union of cycles, and at least one of them is odd. If the weight of  $P$  is less than 1, then we have found an odd cycle of weight less than 1. On the other hand, if for every node  $u$  the weight of a shortest path from  $u'$  to  $u''$  is at least 1, then there is no violated odd cycle inequality.

Since the arc-weights could be negative, we should apply Bellman–Ford algorithm for finding a shortest path, see [1]. We have to see that this graph has no negative cycle.

**Lemma 2.** *The auxiliary graph has no cycle of negative weight.*

**Proof.** Let  $(u, v)$  and  $(v, w)$  be two consecutive arcs in a cycle  $C$ . It follows from (2) and (4) that  $\bar{x}(u, v) + \bar{x}(v, w) \leq 1$ . This implies

$$\bar{x}(C) \leq \frac{|C|}{2}. \quad (9)$$

If  $C$  is a cycle of negative weight, we have

$$|C| - 2\bar{x}(C) < 0,$$

or

$$\bar{x}(C) > \frac{|C|}{2},$$

this contradicts (9).  $\square$

Since each shortest path computation takes  $O(|V|^3)$  time, the entire separation algorithm takes  $O(|V|^4)$  time.

Remark that the number of odd cycles in a  $Y$ -free graph is polynomially bounded. Indeed in a  $Y$ -free graph no two cycles can intersect, so an arc can belong to at most one cycle. Hence for this class of graphs the system defined by (2)–(6) and (7) has a polynomial number of inequalities.

### 3. The extreme points of $P_p(G)$ , when $G$ is a $Y$ -free graph

In this section we give a description of the extreme points of  $P_p(G)$ , when  $G$  is a  $Y$ -free graph. Let  $G = (V, E)$  be a  $Y$ -free graph. Let  $z = (x, y) \in P_p(G)$ . Let  $E_z = \{(u, v) \in E : 0 < z(u, v) < 1\}$  and denote by  $G_z = (V_z, E_z)$  the subgraph of  $G$  induced by  $E_z$ . Denote by  $V_z^<$  the set of nodes  $v$  with  $z(v) > 0$  such that either  $z(u, v) < z(v)$  for  $(u, v) \in E_z$  or  $|\delta_{G_z}^-(v)| = 0$  or  $z(v) = 1$ . Call a node  $v$  with  $z(v) = 1$  a *pendent node*. A directed path from  $v$  to  $w$  is denoted by  $P_w^v$ . Define the size of  $P_w^v$  to be the number of its inner nodes (nodes different from  $v$  and  $w$ ). If the size of  $P_w^v$  is even (resp. odd), we say that  $P_w^v$  is an *even path* (resp. *odd path*). Two paths are said to be *node-disjoint*, if the sets of their inner nodes are disjoint.

For a solution  $z = (x, y)$  and a path  $P_{v_k}^{v_1} = v_1, \dots, v_k$ , define  $z[P_{v_k}^{v_1}]^\delta \in \mathbb{R}^{|E|+|V|}$  to be:

$$\begin{aligned} z[P_{v_k}^{v_1}]^\delta(v_1) &= z(v_1) & \text{if } z(v_1) = 0, \\ z[P_{v_k}^{v_1}]^\delta(v_1) &= z(v_1) - \delta & \text{if } z(v_1) > 0, \\ z[P_{v_k}^{v_1}]^\delta(v_i) &= z(v_i) + (-1)^i \delta & \text{for } i = 2, \dots, k-1 \\ z[P_{v_k}^{v_1}]^\delta(v_k) &= z(v_k), \\ z[P_{v_k}^{v_1}]^\delta(v_i, v_{i+1}) &= z(v_i, v_{i+1}) + (-1)^{i+1} \delta & \text{for } i = 1, \dots, k-1, \end{aligned}$$

and  $z[P_w^v]^\delta(u, v) = z(u, v)$ ,  $z[P_w^v]^\delta(u) = z(u)$ , for all other arcs and nodes (not in the path).

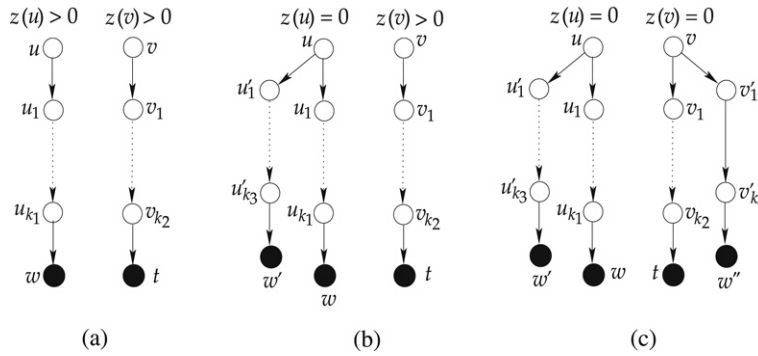


Fig. 3. The three cases of Lemma 6.

Let  $G_z^1, \dots, G_z^q$  be the connected components of  $G_z$ . In the following, we shall study the structure of these connected components when  $z$  is a fractional extreme point of  $P_p(G)$ . For this purpose, let us consider  $G_z^1 = (V_z^1, E_z^1)$ .

**Remark 3.** For every node  $v$  of  $G_z^1$  that is not a pendent node, we have  $|\delta_{G_z^1}^-(v)| \leq 1$ . Moreover, if  $|\delta_{G_z^1}^-(v)| = 1$  then,  $0 < z(v) < 1$ .

This remark is used implicitly in all the proofs of this section. Let  $z = (x, y)$  be an extreme fractional point of  $P_p(G)$ .

**Lemma 4.**  $G_z^1$  does not contain an odd path  $P_w^v$  with  $v, w \in V_z^<$ .

**Proof.** Suppose that  $P_w^v = v, v_1, v_2, \dots, v_k, w$  is such a path, with  $k$  odd. Then the same constraints that are tight for  $z$  are also tight for  $z[P_w^v]^{-\epsilon}$ . This contradicts the fact that  $z$  is an extreme point of  $P_p(G)$ .  $\square$

**Lemma 5.**  $G_z^1$  cannot contain two node-disjoint paths  $P_u^v$  and  $P_w^v$  having the same parity, where  $u, w$  are in  $V_z^<$  and are not necessarily different.

**Proof.** Let  $P_u^v = v, u_1, u_2, \dots, u_{k_1}, u$  and  $P_w^v = v, w_1, w_2, \dots, w_{k_2}, w$  be two node-disjoint paths such that  $k_1$  and  $k_2$  are of the same parity. Let  $z_1 = z[P_u^v]^\epsilon$  and  $z_2 = z_1[P_w^v]^{-\epsilon}$ . Then the same constraints that are tight for  $z$  are also tight for  $z_2$ , a contradiction.  $\square$

**Lemma 6.** Let  $w, t \in V_z^1$  be two non-necessarily different pendent nodes. If  $G_z^1$  contains two node-disjoint paths  $P_w^u$  and  $P_t^v$ , such that there is no saturated arc directed into  $u$  and  $v$ , then  $z(u) = z(v) = 0$  and  $u = v$ .

**Proof.** Let  $P_w^u = u, u_1, u_2, \dots, u_{k_1}, w$  and  $P_t^v = v, v_1, v_2, \dots, v_{k_2}, t$ . Three cases are distinguished, as described by Fig. 3, (a)  $u$  and  $v$  in  $V_z^<$ , (b)  $v \in V_z^<$  and  $z(u) = 0$  and (c)  $z(u) = z(v) = 0$  and  $u \neq v$ .

- (a) From Lemma 4, these two paths are even. And from Lemma 5,  $u \neq v$ . Let  $z_1 = z[P_w^u]^\epsilon$ . Then the same constraints that are tight for  $z$  are also tight for  $z_1[P_t^v]^{-\epsilon}$ , a contradiction.
- (b) By definition, there must exist a path  $P_{w'}^u = u, u'_1, \dots, u'_{k_3}, w'$  from  $u$  to a pendent node  $w'$ ,  $w'$  may coincide with  $w$  and  $t$ , (see Fig. 3(b)). By Lemma 5,  $k_3$  and  $k_1$  are of different parity, and by Lemma 4,  $k_2$  is even. Let  $z_1 = z[P_w^u]^\epsilon$ ,  $z_2 = z_1[P_w^u]^{-\epsilon}$ ,  $z_3 = z_2[P_t^v]^{-\epsilon}$ , and  $z_4 = z_2[P_t^v]^\epsilon$ . Then the same constraints that are tight for  $z$  are also tight for either  $z_3$  or  $z_4$ , again a contradiction.
- (c) There must exist paths  $P_{w'}^u = u, u'_1, \dots, u'_{k_3}, w'$  and  $P_{w''}^v = v, v'_1, \dots, v'_{k_4}, w''$ , where  $w'$  and  $w''$  are pendent nodes,  $w, t, w'$  and  $w''$  are not necessarily different, (see Fig. 3(c)). From Lemma 5,  $k_1$  with  $k_3$  and  $k_2$  with  $k_4$  are of different parity. Again we are going to construct a new vector  $z^*$  such that all constraints that are tight for  $z$  are also tight for  $z^*$ . Suppose for simplicity that  $k_3$  and  $k_2$  are odd and that  $k_1$  and  $k_4$  are even. The other cases may be treated similarly. Let  $z_1 = z[P_w^u]^\epsilon$ ,  $z_2 = z_1[P_w^u]^{-\epsilon}$ ,  $z_3 = z_2[P_t^v]^{-\epsilon}$ , finally  $z_3[P_{w''}^v]^\epsilon$  has the desired property.  $\square$

**Lemma 7.** Let  $C = \{(v_i, v_{i+1}) \mid i = 1, \dots, 2l\} \cup \{(v_{2l+1}, v_1)\}$  be an odd cycle of  $G_z^1$ . Then at most one arc of  $C$  is not saturated.

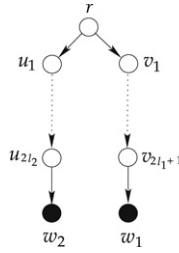


Fig. 4. Two node-disjoint paths,  $P_{w_1}^r$  and  $P_{w_2}^r$ , where  $k_1 = 2l_1 + 1$ ,  $k_2 = 2l_2$ ,  $P_{w_1}^r$  may be empty, and  $w_1$  may coincide with  $w_2$ .

**Proof.** Suppose we have two arcs  $(v_i, v_{i+1})$  and  $(v_j, v_{j+1})$  not saturated, that is,

$$z(v_i, v_{i+1}) < z(v_{i+1}), \quad \text{and}$$

$$z(v_j, v_{j+1}) < z(v_{j+1}),$$

so  $v_{i+1}$  and  $v_{j+1}$  are in  $V_z^<$ . And since  $C$  is odd then, either  $P_{v_{i+1}}^{v_{i+1}}$  or  $P_{v_{j+1}}^{v_{j+1}}$  is odd. This contradicts Lemma 4.  $\square$

We say that  $r \in V_z^1$  is a *root* if  $|\delta_{G_z^1}^-(r)| = 0$ .

**Lemma 8.** If  $G_z^1$  contains a root  $r$  then it contains a directed path from  $r$  to a pendent node.

**Proof.** Starting from  $r$  perform a depth-first search for a pendent node using the arcs of  $G_z^1$ . Since there is no  $Y$  the search should end on a pendent node.  $\square$

**Lemma 9.** If  $G_z^1$  contains a root then it contains exactly one.

**Proof.** Let  $r_1$  and  $r_2$  be two roots in  $G_z^1$ . Since  $G_z^1$  has no  $Y$ , there is a directed path  $P_{w_1}^{r_1}$  from  $r_1$  to a pendent node  $w_1$ . Also there is a directed path  $P_{w_2}^{r_2}$  from  $r_2$  to a pendent node  $w_2$ . These two paths are disjoint because there is no  $Y$ . Lemma 6 implies  $r_1 = r_2$ .  $\square$

**Lemma 10.** In  $G_z^1$  there is a path from  $r$  to any pendent node.

**Proof.** Consider a pendent node  $v$  in  $G_z^1$ . There is an undirected path  $P$  from  $r$  to  $v$ . Let  $P_1$  be the maximal directed path directed away from  $r$  using only arcs in  $P$ . Let  $w$  be the last node in  $P_1$ , since there is no  $Y$ ,  $w$  is a pendent node. If  $w = v$  we are done, otherwise let  $P_2$  be the maximal path directed into  $v$ , included in  $P$ . Let  $s$  be the first node in  $P_2$ . Let  $P_3$  be the other maximal path directed away from  $s$  included in  $P$ . Since there is no  $Y$ , the last node in  $P_3$  is pendent. From Lemma 5, we have that  $P_2$  and  $P_3$  have different parities. Let  $z_1 = z[P_2]^\epsilon$ ,  $z_2 = z_1[P_3]^{-\epsilon}$ . If  $z(r) > 0$ , then from Lemma 4,  $P_1$  must be even. Hence  $z_2[P_1]^\epsilon$  or  $z_2[P_1]^{-\epsilon}$  is a vector that satisfies as equation the same constraints as  $z$ .

Otherwise, if  $z(r) = 0$ , then there must be a path  $P'_1$  from  $r$  to pendent node, where  $P_1$  and  $P'_1$  are node-disjoint. From Lemma 5,  $P_1$  and  $P'_1$  have different parities. Define  $z_3 = z_2[P_1]^\epsilon$  and  $z_4 = z_2[P_1]^{-\epsilon}$ , then either  $z_3[P'_1]^{-\epsilon}$  or  $z_4[P'_1]^\epsilon$  is a vector that satisfies as equation the same constraints as  $z$ .  $\square$

Now we have to study two cases as follows.

**Case 1.**  $G_z^1$  does not contain a directed cycle.

In this case,  $G_z^1$  must contain a root  $r$ . From Lemma 9, the root  $r$  is unique. From Lemma 5,  $|\delta_{G_z^1}^+(r)| \leq 2$ . Otherwise, since  $G_z^1$  is  $Y$ -free, there must exist two node-disjoint paths having the same parity from  $r$  to two pendent nodes  $w_1$  and  $w_2$ .

**Proposition 11.**  $G_z^1$  consists either of

- two paths  $P_{w_1}^r = r, v_1, \dots, v_{k_1}, w_1$  and  $P_{w_2}^r = r, u_1, \dots, u_{k_2}, w_2$  of different parity, or
- just the even path  $P_{w_2}^r = r, u_1, \dots, u_{k_2}, w_2$ , see Fig. 4.

Moreover, all the arcs, different from  $(v_{k_1}, w_1)$  and  $(u_{k_2}, w_2)$ , are saturated by  $z$ , and

$$\sum_{i=1}^{k_1} z(v_i) + \sum_{i=1}^{k_2} z(u_i) + z(r) = l_1 + l_2 + z(u_{k_2}), \text{ if } G_z^1 \text{ consists of two paths,} \quad (10)$$

$$\sum_{i=1}^{k_2} z(u_i) + z(r) = l_2 + z(u_{k_2}), \text{ if } G_z^1 \text{ consists of an even path.} \quad (11)$$

**Proof.** If  $|\delta^+(r)| = 2$ , then there are two node-disjoint paths to some pendent nodes. If  $|\delta^+(v)| \geq 2$  for a node  $v$  in any of these paths, then there are two node-disjoint paths  $P_1$  and  $P_2$  from  $v$  to some pendent nodes. Lemma 5 implies that  $P_1$  and  $P_2$  have different parities. This implies that there are two node-disjoint paths of the same parity from  $r$  to some pendent nodes, a contradiction.

If  $|\delta^+(r)| = 1$ , then there is a directed path from  $r$  to a pendent node. If  $|\delta^+(v)| \geq 2$  for a node  $v$  in this path, then there are two node-disjoint paths  $P_1$  and  $P_2$  from  $v$  to some pendent nodes. Lemma 5 implies that  $P_1$  and  $P_2$  have different parities. In this case  $z(r) > 0$ , thus Lemma 4 yields a contradiction.

Let  $S$  be the set of nodes in  $V_z^1$  that do not belong to any path from  $r$  to a pendent node. Since  $G_z^1$  is connected, there must exist a node  $s \in S$  incident to one of the nodes in a path from  $r$  to a pendent node. Since  $G$  is  $Y$ -free, we must have an arc  $(s, w)$  and  $w$  is a pendent node. Since  $s$  is not a root, there must exist an arc directed into  $s$ , repeating this process we must end up with the root  $r$ , which is a contradiction since  $s$  does not belong to any path from  $r$  to a pendent node, or we must have a directed cycle, which is impossible.

We have exactly one root and  $|\delta_{G_z^1}^+(v)| = 1$  for all non-pendent nodes  $v \in V_z^1 \setminus \{r\}$ . It follows that, if  $|\delta_{G_z^1}^+(r)| = 2$ , then  $G_z^1$  consists of two node-disjoint paths from  $r$  to two pendent nodes or to the same pendent node. From Lemma 5, these two paths must be of different parity. If  $|\delta_{G_z^1}^+(r)| = 1$ ,  $G_z^1$  is a path from  $r$  to a pendent node. Since  $z(r) > 0$ , it follows from Lemma 4 that this path is even.

Let  $k_1 = 2l_1 + 1$  and  $k_2 = 2l_2$ . Remark that if  $|\delta_{G_z^1}^+(r)| = 2$  then  $z(r) = 0$ , otherwise this will contradict Lemma 4. Suppose  $z(v_l, v_{l+1}) < z(v_{l+1})$ , for  $l \in \{1, \dots, k_1 - 1\}$ . So  $v_{l+1} \in V_z^<$ . It follows from Lemma 4 that the path  $P_{w_1}^{l+1}$  has to be even. Since  $k_1$  is odd then, both  $l$  and  $k_2$  are even, which contradicts Lemma 5. In the same manner it may be shown that all the arcs, different from  $(v_{k_1}, w_1)$  and  $(u_{k_2}, w_2)$ , are saturated by  $z$ .

Now, let us verify (10). From the above we have,

$$z(r, v_1) = z(v_1), \quad (12)$$

$$z(r, u_1) = z(u_1), \quad (13)$$

$$z(v_i, v_{i+1}) = z(v_{i+1}) \quad \text{for } i = 1, \dots, k_1 - 1, \quad (14)$$

$$z(u_i, u_{i+1}) = z(u_{i+1}) \quad \text{for } i = 1, \dots, k_2 - 1, \quad (15)$$

and equalities (2) with respect to  $r$ ,  $v_i$ ,  $i = 1, \dots, k_1 - 1$ , and  $u_i$ ,  $i = 1, \dots, k_2 - 1$ , give

$$z(r, v_1) + z(r, u_1) = 1 - z(r), \quad (16)$$

$$z(v_i, v_{i+1}) = 1 - z(v_i) \quad \text{for } i = 1, \dots, k_1 - 1, \quad (17)$$

$$z(u_i, u_{i+1}) = 1 - z(u_i) \quad \text{for } i = 1, \dots, k_2 - 1. \quad (18)$$

The combination of Eq. (14) with (17) and (15) with (18) gives,

$$z(v_{i+1}) = 1 - z(v_i) \quad \text{for } i = 1, \dots, k_1 - 1, \quad (19)$$

$$z(u_{i+1}) = 1 - z(u_i) \quad \text{for } i = 1, \dots, k_2 - 1. \quad (20)$$

The sum of Eqs. (12)–(15) is equal to the sum of Eqs. (16)–(18), hence

$$\sum_{i=1}^{k_1} z(v_i) + \sum_{i=1}^{k_2} z(u_i) = k_1 + k_2 - 1 - z(r) - \sum_{i=1}^{k_1-1} z(v_i) - \sum_{i=1}^{k_2-1} z(u_i). \quad (21)$$

Recall that  $k_1 = 2l_1 + 1$  and  $k_2 = 2l_2$ . Now by considering (19) and (20), Eq. (21) may be rewritten as follows

$$\sum_{i=1}^{k_1} z(v_i) + \sum_{i=1}^{k_2} z(u_i) + z(r) = 2l_1 + 2l_2 - l_1 - l_2 + z(u_{k_2}),$$

hence

$$\sum_{i=1}^{k_1} z(v_i) + \sum_{i=1}^{k_2} z(u_i) + z(r) = l_1 + l_2 + z(u_{k_2}).$$

Eq. (11) can be obtained in a similar way.  $\square$

**Case 2.**  $G_z^1$  contains a directed cycle  $C = \{(v_i, v_{i+1}) \mid i = 1, \dots, k-1\} \cup \{(v_k, v_1)\}$ .

If  $k = 2l$  consider the vector  $\bar{z}$  defined below. The same constraints that are tight for  $z$  are also tight for  $\bar{z}$ .

$$\begin{aligned} \bar{z}_{v_i} &= z(v_i) + (-1)^i \epsilon & \text{for } i = 1, \dots, k, \\ \bar{z}(v_i, v_{i+1}) &= z(v_i, v_{i+1}) + (-1)^{i+1} \epsilon & \text{for } i = 1, \dots, k-1, \\ \bar{z}(v_k, v_1) &= z(v_k, v_1) - \epsilon, \end{aligned}$$

and  $\bar{z}(u, v) = z(u, v)$ ,  $\bar{z}(u) = z(u)$  for all other nodes and arcs not in  $C$ .

So we can assume that  $C$  is odd. Let  $k = 2l + 1$  and  $V(C) = \{v_1, \dots, v_{2l+1}\}$ . From Lemma 7, we have to consider two subcases:

**Case 2.1.**  $C$  contains exactly one non-saturated arc.

**Proposition 12.** We should have that  $\delta_{G_z^1}^+(V(C)) = \emptyset$ , and if  $(v_1, v_2)$  is the non-saturated arc then

$$\sum_{i=1}^{2l+1} z(v_i) = l + z(v_2). \quad (22)$$

**Proof.** Note that  $v_2 \in V_z^<$ . Let  $v_m$  be a node of  $V(C)$  such that  $|\delta_{G_z^1}^+(v_m)| \geq 2$ ,  $v_m$  may coincide with  $v_1$  or  $v_2$ . Since  $G$  is  $Y$ -free, there is a path  $P_w^{v_m}$  having all its nodes not in  $V(C)$  with  $w$  a pendent node. We have four cases to consider,  $P_w^{v_m}$  is odd or even, and  $m$  is odd or even. Suppose that  $P_w^{v_m}$  is odd, either  $m$  is odd, and in this case the paths  $P_{v_2}^{v_m}$  and  $P_w^{v_m}$  are both odd, which contradicts Lemma 5, or  $m$  is even and in this case the path defined by the junction of  $P_{v_m}^{v_2}$  and  $P_w^{v_m}$  is odd, which contradicts Lemma 4. The same arguments hold when  $P_w^{v_m}$  is even. So we have proved that  $\delta^+(V(C)) = \emptyset$ .

From Eq. (1),

$$\begin{aligned} z(v_i, v_{i+1}) &= 1 - z(v_i) & \text{for } i = 2, \dots, 2l \\ z(v_{2l+1}, v_1) &= 1 - z(v_{2l+1}). \end{aligned}$$

Also,  $z(v_i, v_{i+1}) = z(v_{i+1})$  for each arc of  $C$  different from  $(v_1, v_2)$ . The combination of these equations gives

$$\sum_{i=2}^{2l+1} (1 - z(v_i)) = \sum_{i=1, i \neq 2}^{2l+1} z(v_i) \quad (23)$$

we also have,

$$\begin{aligned} z(v_{i+1}) &= 1 - z(v_i) & \text{for } i = 2, \dots, 2l \\ z(v_1) &= 1 - z(v_{2l+1}), \end{aligned}$$

these imply that  $(1 - z(v_i)) + (1 - z(v_{i+1})) = 1$ , for  $i = 1, \dots, 2l$ . Hence the Eq. (23) is equivalent to

$$\sum_{i=1, i \neq 2}^{2l+1} z(v_i) = l. \quad \square$$



**Case 2.2.** All the arcs of  $C$  are saturated.

Note that in this case, if one of the node-variables or arc-variables of  $C$  has a value  $1/2$ , then all the other node-variables and arc-variables have the same value and that  $G_z^1$  consists of only the cycle  $C$ . In general, the structure of  $G_z^1$  is described by the following proposition, see the figure below.

**Proposition 13.** *We have that  $|\delta_{G_z^1}^+(V(C))| \leq 1$ . And if  $|\delta_{G_z^1}^+(V(C))| = 1$ , then  $G_z^1$  consists of the odd cycle  $C$  and a path  $P_w^{v_m} = v_m, u_1, \dots, u_k, w$ , going from a node  $v_m \in V(C)$  to a pendent node  $w$ ,  $m \in \{1, \dots, 2l+1\}$ . Moreover, all the arcs different from  $(u_k, w)$  are saturated and*

$$\sum_{i=1}^{2l+1} z(v_i) + \sum_{i=1}^k z(u_i) = l + l_1 + z(v_m) \quad \text{if } k = 2l_1, \quad (24)$$

$$\sum_{i=1}^{2l+1} z(v_i) + \sum_{i=1}^k z(u_i) = l + l_1 + 1 - z(v_m, v_{m+1}) \quad \text{if } k = 2l + 1. \quad (25)$$

**Proof.** Suppose that  $|\delta_{G_z^1}^+(V(C))| \geq 2$ . Let  $(v_{i_1}, u)$  and  $(v_{i_2}, u')$  be two arcs in  $\delta_{G_z^1}^+(V(C))$ . We must have two node-disjoint paths:  $P_1$  from  $v_{i_1}$  to a pendent node  $w_1$ , containing  $(v_{i_1}, u)$ , and  $P_2$  from  $v_{i_2}$  to a pendent node  $w_2$ , containing  $(v_{i_2}, u')$ . If  $i_1 \neq i_2$ , then one may define two node-disjoint paths having the same parity, going from  $v_{i_1}$  (or  $v_{i_2}$ ), to  $w_1$  and  $w_2$ , which contradicts Lemma 5. If  $i_1 = i_2$ , then by Lemma 5,  $P_1$  and  $P_2$  are of different parity. For simplicity let  $i_1 = i_2 = 1$ . Let  $P'_1$  and  $P'_2$  be the portions of  $P_1$  and  $P_2$ , from  $u$  to  $w_1$  and from  $u'$  to  $w_2$ , respectively. Define  $z_1 = z[P'_1]^{+\epsilon}$ ,  $z_2 = z_1[P'_2]^{+\epsilon}$ , and  $\bar{z}$  to be

$$\begin{aligned} \bar{z}(v_1, u) &= z_2(v_1, u) - \epsilon, \\ \bar{z}(v_1, u') &= z_2(v_1, u') - \epsilon, \\ \bar{z}(v_1) &= z_2(v_1) + \epsilon, \\ \bar{z}(v_1, v_2) &= z_2(v_1, v_2) + \epsilon, \\ \bar{z}(v_i) &= z_2(v_i) + (-1)^i \epsilon \quad \text{for } i = 2, \dots, 2l+1, \\ \bar{z}(v_i, v_{i+1}) &= z_2(v_i, v_{i+1}) + (-1)^{i+1} \epsilon \quad \text{for } i = 2, \dots, 2l, \\ \bar{z}(v_{2l+1}, v_1) &= z_2(v_{2l+1}, v_1) + \epsilon, \end{aligned}$$

and  $\bar{z}(u) = z_2(u)$  (resp.  $\bar{z}(u, v) = z_2(u, v)$ ) for all other nodes (resp. arcs).

Since  $P_1$  and  $P_2$  are of different parity,  $\bar{z}$  also satisfies (3). We have that the constraints that are tight for  $z$  are also tight for  $\bar{z}$ , which contradicts the fact that  $z$  is an extreme point of  $P_p(G)$ .

Thus it may be assumed that  $|\delta_{G_z^1}^+(V(C))| = 1$ . Let  $(v_m, u_1)$  be the unique arc of  $\delta_{G_z^1}^+(V(C))$ . For simplicity, take  $m = 1$ . There must exist a path from  $v_1$  to a pendent node, call it  $P_w^{v_1} = v_1, u_1, \dots, u_k, w$ . Suppose there exists a node  $u_r$ ,  $1 \leq r \leq k$ , with  $|\delta_{G_z^1}^+(u_r)| \geq 2$ , then there are two node-disjoint paths  $P_1$  and  $P_2$  from  $u_r$  to some pendent nodes.

Lemma 5 implies that  $P_1$  and  $P_2$  have different parities. Suppose  $r$  is odd. Define  $z_1 = z[P_{u_r}^{u_1}]^{-2\epsilon}$ ,  $z_2 = z_1[P_1]^{-\epsilon}$ ,  $z_3 = z_2[P_2]^{-\epsilon}$ . Define  $\bar{z}$  to be

$$\begin{aligned} \bar{z}(v_1, u_1) &= z_3(v_1, u_1) + 2\epsilon, \\ \bar{z}(v_1) &= z_3(v_1) - \epsilon, \\ \bar{z}(v_1, v_2) &= z_3(v_1, v_2) - \epsilon, \\ \bar{z}(v_i) &= z_3(v_i) + (-1)^{i+1} \epsilon \quad \text{for } i = 2, \dots, 2l+1, \\ \bar{z}(v_i, v_{i+1}) &= z_3(v_i, v_{i+1}) + (-1)^i \epsilon \quad \text{for } i = 2, \dots, 2l, \\ \bar{z}(v_{2l+1}, v_1) &= z_3(v_{2l+1}, v_1) - \epsilon, \end{aligned}$$

and  $\bar{z}(u) = z_3(u)$  (resp.  $\bar{z}(u, v) = z_3(u, v)$ ), for all other nodes (resp. arcs). Thus  $\bar{z}$  and  $z$  satisfy the same constraints as equations, this contradicts again the fact that  $z$  is an extreme point. Using the same ideas, one obtains the same contradiction when  $r$  is even.

Thus, it may be assumed that  $G_z^1$  consists of only the cycle  $C$  and the path  $P_w^{v_1}$ . Suppose  $(u_r, u_{r+1})$  is non-saturated by  $z$ , for  $0 \leq r \leq k-1$ , where  $u_0 = v_1$ . It follows from Lemma 4 that  $P_w^{u_{r+1}}$  is even.

Define  $z_1 = z[P_{u_{r+1}}^{u_1}]^{-2\epsilon}$ ,  $z_2 = z_1[P_w^{u_{r+1}}]^{-\epsilon}$  if  $P_{u_{r+1}}^{u_1}$  is odd, otherwise  $z_2 = z_1[P_w^{u_{r+1}}]^{+\epsilon}$  and let  $\bar{z}$  to be

$$\begin{aligned}\bar{z}(v_1, u_1) &= z_2(v_1, u_1) + 2\epsilon, \\ \bar{z}(v_1) &= z_2(v_1) - \epsilon, \\ \bar{z}(v_1, v_2) &= z_2(v_1, v_2) - \epsilon, \\ \bar{z}(v_i) &= z_2(v_i) + (-1)^{i+1}\epsilon \quad \text{for } i = 2, \dots, 2l+1, \\ \bar{z}(v_i, v_{i+1}) &= z_2(v_i, v_{i+1}) + (-1)^i\epsilon \quad \text{for } i = 2, \dots, 2l, \\ \bar{z}(v_{2l+1}, v_1) &= z_2(v_{2l+1}, v_1) - \epsilon,\end{aligned}$$

and  $\bar{z}(u) = z_2(u)$  (resp.  $\bar{z}(u, v) = z_2(u, v)$ ), for all other nodes (resp. arcs). We have that constraints (2)–(6) that are tight for  $z$  are also tight for  $\bar{z}$ , which contradicts the fact that  $z$  is an extreme point.

Thus every arc, different from  $(u_k, w)$ , is saturated, so

$$z(v_1, u_1) = z(u_1), \quad (26)$$

$$z(u_{i-1}, u_i) = z(u_i) \quad \text{for } i = 2, \dots, k, \quad (27)$$

$$z(v_i, v_{i+1}) = z(v_{i+1}) \quad \text{for } i = 1, \dots, 2l, \quad (28)$$

$$z(v_{2l+1}, v_1) = z(v_1). \quad (29)$$

Now we have to see that (24) and (25) hold. Eq. (2) with respect to  $u_i$ ,  $i = 1, \dots, k-1$ , and  $v_i$ ,  $i = 1, \dots, 2l+1$ , gives

$$z(v_1, u_1) + z(v_1, v_2) = 1 - z(v_1), \quad (30)$$

$$z(v_i, v_{i+1}) = 1 - z(v_i) \quad \text{for } i = 2, \dots, 2l, \quad (31)$$

$$z(v_{2l+1}, v_1) = 1 - z(v_{2l+1}), \quad (32)$$

$$z(u_i, u_{i+1}) = 1 - z(u_i) \quad \text{for } i = 1, \dots, k-1. \quad (33)$$

The combination of Eqs. (26)–(29) with (30)–(33) gives,

$$z(u_1) + z(v_2) = 1 - z(v_1), \quad (34)$$

$$z(v_{i+1}) = 1 - z(v_i) \quad \text{for } i = 1, \dots, 2l, \quad (35)$$

$$z(u_{i+1}) = 1 - z(u_i) \quad \text{for } i = 1, \dots, k-1. \quad (36)$$

The sum of Eqs. (26)–(29) is equal to the sum of Eqs. (30)–(33), hence

$$\sum_{i=1}^{2l+1} z(v_i) + \sum_{i=1}^k z(u_i) = (2l+1) - \sum_{i=1}^{2l} z(v_i) - z(v_{2l+1}) + (k-1) - \sum_{i=1}^{k-1} z(u_i). \quad (37)$$

By considering (34)–(36), Eq. (37) may be rewritten as follows.

If  $k = 2l_1$ :

$$\begin{aligned}\sum_{i=1}^{2l+1} z(v_i) + \sum_{i=1}^k z(u_i) &= (2l+1) - (1 - z(u_1)) - (l-1) - z(v_{2l+1}) + (k-1) - z(u_1) - (l_1-1), \\ &= l + l_1 + 1 - z(v_{2l+1}), \\ &= l + l_1 + z(v_1).\end{aligned}$$

If  $k = 2l_1 + 1$ :

$$\begin{aligned}\sum_{i=1}^{2l+1} z(v_i) + \sum_{i=1}^k z(u_i) &= (2l+1) - (1 - z(u_1)) - (l-1) - z(v_{2l+1}) + 2l_1 - l_1, \\ &= l + l_1 + 1 + z(u_1) - z(v_{2l+1}), \\ &= l + l_1 + z(u_1) + z(v_1), \\ &= l + l_1 + 1 - z(v_1, v_2). \quad \square\end{aligned}$$

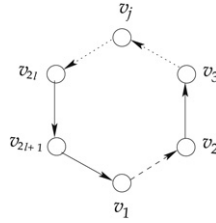


Fig. 5. Odd cycle  $C$ ,  $\delta^+(V(C)) = \emptyset$ . All arcs are saturated except for  $(v_1, v_2)$ .

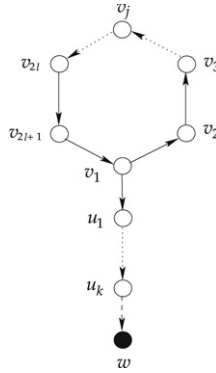


Fig. 6. Odd cycle  $C$ ,  $|\delta^+(V(C))| = 1$ , all arcs are saturated except for  $(u_k, w)$ .

**Theorem 14.** Let  $G$  be a  $Y$ -free graph. Let  $z = (x, y)$  be a fractional extreme point of  $P_p(G)$ . Then the following hold:

- (i)  $G_z$  contains  $q$  connected components,  $G_z^1, \dots, G_z^q$ , with  $q \geq 2$ ,
- (ii)  $G_z$  contains at most one component that corresponds to the graph of Fig. 4. The other components are all odd cycles where each arc is saturated by  $z$ .
- (iii) The values of  $z$  are 0, 1 or  $1/2$ .

**Proof.** (i) If  $G_z$  is connected. Then from Propositions 11–13,  $G_z$  is one of the graphs of Figs. 4–6 or an odd cycle where each node and arc is associated with the value  $1/2$ . These propositions also show that  $\sum_{v \in V_z} z(v)$  is fractional, since  $z(v) = 0$  or  $1$  for every  $v \notin V_z$ , we have that  $\sum_{v \in V} z(v)$  is fractional, which is impossible.

(ii) From Propositions 11–13, any connected component of  $G_z$  is one of the graphs of Figs. 4–6 or an odd cycle where each node and arc is associated with the value  $1/2$ . Let  $G_z^1$  and  $G_z^2$  be two connected components of  $G_z$ . Suppose  $G_z^1$  is the graph of Fig. 4. Let  $G_z^2$  be another connected component of  $G_z$  that corresponds to Fig. 5. Recall that  $G_z^1$  consists of two node-disjoint paths  $P_{w_1}^r$  and  $P_{w_2}^r$  having different parities and  $G_z^2$  consists of an odd cycle  $C = \{(v_i, v_{i+1}) \mid i = 1, \dots, 2l\} \cup \{(v_{2l+1}, v_1)\}$ , where  $(v_1, v_2)$  is non-saturated by  $z$ . Define recursively a new vector  $\bar{z}$  as follows.  $z_1 = z[P_{w_1}^r]^{-\epsilon}$ ,  $z_2 = z_1[P_{w_2}^r]^{+\epsilon}$ ,  $\bar{z} = z_2[P_{v_2}^{v_2}]^{-\epsilon}$ . Any arc saturated by  $z$  remains saturated by  $\bar{z}$ . Moreover, Eq. (3) holds.

$$\begin{aligned} \sum_{v \in V} \bar{z}(v) &= \sum_{v \in V_z^1} \bar{z}(v) + \sum_{v \in V_z^2} \bar{z}(v) + \sum_{v \in V \setminus (V_z^1 \cup V_z^2)} \bar{z}(v) \\ &= \sum_{v \in V_z^1} z(v) - \epsilon + \sum_{v \in V_z^2} z(v) + \epsilon + \sum_{v \in V \setminus (V_z^1 \cup V_z^2)} z(v) \\ &= \sum_{v \in V} z(v) = p. \end{aligned}$$

We conclude that every constraint that is tight for  $z$  is also tight for  $\bar{z}$ , which contradicts the fact that  $z$  is an extreme point. The cases where  $G_z^2$  corresponds to Fig. 4 or Fig. 6 may be treated similarly.

Now suppose that  $G_z^1$  is the graph of Fig. 5 or Fig. 6, that is  $G_z^1$  is an odd cycle  $C = \{(v_i, v_{i+1}) \mid i = 1, \dots, 2l\} \cup \{(v_{2l+1}, v_1)\}$  and  $(v_1, v_2)$  is non-saturated by  $z$  or an odd cycle  $C = \{(v_i, v_{i+1}) \mid i = 1, \dots, 2l\} \cup \{(v_{2l+1}, v_1)\}$  and

a path  $P_w^{v_m} = v_m, u_1, \dots, u_k, w$  for  $v_m \in V(C)$ , respectively. If  $G_z^2$  is one of the graphs of Figs. 4 and 5 or Fig. 6, then as above we can define a solution  $\bar{z}$  such that the same constraints that are tight for  $z$  are also tight for  $\bar{z}$ . Thus all other connected components,  $G_z^2, \dots, G_z^q$  of  $G_z$  consist of an odd cycle where each arc is saturated and that  $z(v) = 1/2$  for all  $v \in (V_z \setminus V_z^1)$ . In both cases, Propositions 12 and 13 imply that  $\sum_{v \in V_z^1} z(v)$  must be fractional. Hence  $q - 1$  must be odd, and since  $\sum_{v \in V_z} z(v)$  is an integer, if  $G_z^1$  corresponds to Fig. 5 by Proposition 12 we have that  $z(v_2) = 1/2$  and if  $G_z^1$  corresponds to Fig. 6 by Proposition 13 we have  $z(v_m) = 1/2$  if  $k$  is even, otherwise  $z(v_m, v_{m+1}) = 1/2$ . In both cases, all the arcs of  $C$  are saturated and thus  $G_z^1$  is also an odd cycle where each arc is saturated.

(iii) By the definition of  $G_z$ ,  $z$  takes the values 0 or 1 for any arc or node not in  $G_z$ . Now (iii) is a straightforward consequence of (ii) and Propositions 11–13.  $\square$

**Corollary 15.** *If  $G$  is a  $Y$ -free graph without odd cycles, then  $pMP(G)$  is completely described by constraints (2)–(6). That is,  $PC_p(G)$  is integral.*

#### 4. Description of $pMP(G)$ , when $G$ is a $Y$ -free graph

We show that the addition of the odd cycle inequalities (7) to (2)–(6), completely describes  $pMP(G)$  when  $G$  is a  $Y$ -free graph.

Call  $PC_p(G)$  the polytope described by constraints (2)–(6) and inequalities (7).

**Theorem 16.** *If  $G$  is a  $Y$ -free graph then,  $pMP(G) = PC_p(G)$ .*

**Proof.** The proof is by induction on the number of arcs. Obviously, the theorem is true for small graphs, with no more than 2 arcs. Suppose it is true for any  $Y$ -free graph with no more than  $m$  arcs and let  $G$  contain exactly  $m + 1$  arcs. Suppose  $pMP(G) \neq PC_p(G)$ , and let  $z = (x, y)$  be a fractional extreme point of  $PC_p(G)$ .

**Claim 1.**  $0 < z(u, v) < 1$  for all  $(u, v) \in E$ .

**Proof.** (i) Let  $(u, v) \in E$  with  $z(u, v) = 0$ . Let  $G' = (V, E \setminus (u, v))$ , and  $z'$  the restriction of  $z$  to  $G'$ . It is clear that  $z' \in PC_p(G')$ . Suppose that  $z' = 1/2z_1 + 1/2z_2$ , where  $z_1, z_2 \in PC_p(G')$ ,  $z_1 \neq z_2$ . Let  $\bar{z}_1$  (resp.  $\bar{z}_2$ ) be the vector obtained by adding a zero component to  $z_1$  (resp.  $z_2$ ). We have that  $z = 1/2\bar{z}_1 + 1/2\bar{z}_2$ .

Now let us see that  $\bar{z}_1$  and  $\bar{z}_2$  are in  $PC_p(G)$ . Clearly they satisfy (2)–(6), so we just have to see that constraints (7) are satisfied. Consider the odd cycle

$$C = \{(w_i, w_{i+1}) \mid i = 1, \dots, 2l\} \cup \{(u, v)\},$$

where  $u = w_{2l+1}$  and  $v = w_1$ . We have that

$$z_1(w_{2i-1}, w_{2i}) + z_1(w_{2i}, w_{2i+1}) \leq 1, \quad \text{for } i = 1, \dots, l.$$

This implies  $\bar{z}_1(C) \leq l$ . The same is true for  $\bar{z}_2$ . Therefore  $\bar{z}_1$  and  $\bar{z}_2$  are in  $PC_p(G)$ .

We then have a contradiction because  $z$  is an extreme point. So  $z'$  must be an extreme point of  $PC_p(G')$  and because of the induction hypothesis, it must be integral.

(ii) Let  $(u, v) \in E$  with  $z(u, v) = 1$ . This implies  $z(u) = 0$ . From (i) we have that  $u$  is incident only to  $(u, v)$ . Let  $G' = (V \setminus \{u\}, E \setminus (u, v))$ , and  $z'$  the restriction of  $z$  to  $G'$ . It is clear that  $z' \in PC_p(G')$ . Suppose that  $z' = 1/2z_1 + 1/2z_2$ , where  $z_1, z_2 \in PC_p(G')$ ,  $z_1 \neq z_2$ . Let  $\bar{z}_1$  (resp.  $\bar{z}_2$ ) be the vector obtained by adding a zero and a one component to  $z_1$  (resp.  $z_2$ ). We have that  $z = 1/2\bar{z}_1 + 1/2\bar{z}_2$ .

First we should see that  $\bar{z}_1$  and  $\bar{z}_2$  are in  $PC_p(G)$ . Clearly they satisfy (2)–(6), and since  $(u, v)$  does not belong to any cycle, constraints (7) are satisfied.

We have then a contradiction because  $z$  is an extreme point. So  $z'$  must be an extreme point of  $PC_p(G')$ , and because of the induction hypothesis it must be integral.  $\square$

In what follows, we will show that  $G$  contains no odd cycle. Let us assume the contrary, and let  $C = \{(v_i, v_{i+1}) \mid i = 1, \dots, 2l\} \cup \{(v_{2l+1}, v_1)\}$  be an odd cycle.

**Remark 17.**  $0 < z(v_i) < 1$ , for  $i = 1, \dots, 2l + 1$ .

This remark follows from [Claim 1](#).

**Remark 18.** Since  $G$  is  $Y$ -free, if  $C_1$  and  $C_2$  are two directed cycles in  $G$ , then  $C_1 \cap C_2 = \emptyset$ .

**Claim 2.** At most one arc of  $C$  is non-saturated by  $z$ .

**Proof.** Suppose that we have two arcs  $(v_i, v_{i+1})$  and  $(v_j, v_{j+1})$  that are not saturated by  $z$ , that is,

$$\begin{aligned} z(v_i, v_{i+1}) &< z(v_{i+1}), \quad \text{and} \\ z(v_j, v_{j+1}) &< z(v_{j+1}). \end{aligned}$$

Since  $C$  is odd, then either  $P_{v_{j+1}}^{v_{i+1}}$  or  $P_{v_{i+1}}^{v_{j+1}}$  is odd. Suppose  $P_{v_{j+1}}^{v_{i+1}}$ . By [Claim 1](#),  $0 < z(u, v) < 1$  for every arc  $(u, v)$  in  $P_{v_{j+1}}^{v_{i+1}}$  and by [Remark 17](#),  $0 < z(v) < 1$  for every node  $v$  in  $V(C)$ . Also by [Remark 18](#), the nodes in  $P_{v_{j+1}}^{v_{i+1}}$  do not belong to another cycle. It follows that  $z[P_{v_{j+1}}^{v_{i+1}}]^{+\epsilon}$  satisfies with equation the same constraints that  $z$  does, which contradicts the fact that  $z$  is an extreme point of  $PC_p(G)$ .  $\square$

**Claim 3.** If  $P_1$  and  $P_2$  are two paths going from  $V(C)$  to some pendent nodes, where all the inner nodes of both  $P_1$  and  $P_2$  are not in  $V(C)$ , then  $P_1$  and  $P_2$  cannot have the same parity.

**Proof.** Let  $P_{u_{l_1}}^{v_k} = v_k, u_1, \dots, u_{l_1}$ ,  $P_{u'_{l_2}}^{v_m} = v_m, u'_1, \dots, u'_{l_2}$ , be two paths having the same parity, with  $z(u_{l_1}) = z(u'_{l_2}) = 1$ ,  $k \leq m$ . Notice that  $(v_k, u_1)$  and  $(v_m, u'_1)$  are in  $\delta^+(V(C))$ , where  $l_1$  and  $l_2$  may be equal to 1, and  $u_{l_1}$ ,  $u'_{l_2}$  may coincide. Remark that from [Claim 1](#), we have  $0 < z(v) < 1$  for any node  $v$  in  $P_1$  and  $P_2$  ( $v$  is not a pendent node).

Let  $P_{u'_{l_2}}^{v_k}$  be the path obtained by joining the path in  $C$  from  $v_k$  to  $v_m$  and  $P_{u'_{l_2}}^{v_m}$  and let  $P_{u_{l_1}}^{v_m}$  be obtained by joining the path in  $C$  from  $v_m$  to  $v_k$  and  $P_{u_{l_1}}^{v_k}$ . Since  $C$  is odd, then either  $P_{u'_{l_2}}^{v_k}$  and  $P_{u_{l_1}}^{v_k}$  or  $P_{u_{l_1}}^{v_m}$  and  $P_{u'_{l_2}}^{v_m}$ , are of the same parity.

Suppose that  $P_{u_{l_1}}^{v_k}$  and  $P_{u_{l_1}}^{v_m}$  have the same parity. Consider  $z_1 = z[P_{u_{l_1}}^{v_k}]^\epsilon$  and  $z_2 = z_1[P_{u_{l_1}}^{v_k}]^{-\epsilon}$ , then  $z_2$  and  $z$  satisfy the same constraints with equality. Notice that since  $G$  is  $Y$ -free, the nodes and arcs in  $C$ ,  $P_{u'_{l_2}}^{v_k}$  and  $P_{u_{l_1}}^{v_k}$  cannot appear in any other cycle. A similar proof can be used if  $P_{u_{l_1}}^{v_m}$  and  $P_{u'_{l_2}}^{v_m}$  have the same parity.  $\square$

From [Claim 2](#), we distinguish two cases:

- (a) All arcs of  $C$  are saturated by  $z$ . By [Claim 3](#) we have that  $|\delta^+(V(C))| \leq 2$ . Suppose that  $|\delta^+(V(C))| = 2$ . We can assume that  $\delta^+(V(C)) = \{(v_1, u), (v_{2l+1}, w)\}$ . We have

$$x(v_{2i-1}, v_{2i}) + x(v_{2i}, v_{2i+1}) = 1,$$

for  $i = 1, \dots, l$ , which implies  $x(C) = l + x(v_{2l+1}, v_1)$ . Thus one of inequalities (7) is violated. If  $|\delta^+(V(C))| \leq 1$  a similar proof can be used.

- (b) There is exactly one arc in  $C$  that is not saturated by  $z$ . First we are going to prove that any path from  $V(C)$  to a pendent node is even.

Let  $(v_{2l+1}, v_1)$  be the non-saturated arc in  $C$ . Let  $P_w^{v_{2k}}$  be an odd path from  $v_{2k}$  to a pendent node  $w$ . Let  $z_1 = z[P_w^{v_{2k}}]^\epsilon$ , and  $z_2 = z_1[P_{v_1}^{v_{2k}}]^{-\epsilon}$ . Here  $P_{v_1}^{v_{2k}}$  denotes the path in  $C$  from  $v_{2k}$  to  $v_1$ . Since  $G$  is  $Y$ -free, the nodes and arcs in  $C$  and  $P_w^{v_{2k}}$  cannot appear in any other cycle. We have that  $z$  and  $z_2$  satisfy the same constraints as equation, a contradiction.

Now let  $P_w^{v_{2k+1}}$  be an odd path from  $v_{2k+1}$  to a pendent node  $w$ . Let  $P$  be the path obtained by joining the path in  $C$  from  $v_1$  to  $v_{2k+1}$  and  $P_w^{v_{2k+1}}$ . Let  $z_1 = z[P]^\epsilon$ , the vectors  $z$  and  $z_1$  satisfy the same constraints as equation.

This shows that any path from  $V(C)$  to a pendent node is even, and by [Claim 3](#) we have that  $|\delta^+(V(C))| \leq 1$ .

If  $\delta^+(V(C)) = \{(v_{2k+1}, w)\}$ , we have that

$$x(v_{2i-1}, v_{2i}) + x(v_{2i}, v_{2i+1}) = 1,$$

for  $i = 1, \dots, l$ , which implies  $x(C) = l + x(v_{2l+1}, v_1)$ . Thus one of inequalities (7) is violated.

If  $\delta^+(V(C)) = \{(v_{2k}, w)\}$ , we have that

$$\begin{aligned} x(v_{2i-1}, v_{2i}) + x(v_{2i}, v_{2i+1}) &= 1, & \text{for } i = 1, \dots, k-1, \\ x(v_{2i}, v_{2i+1}) + x(v_{2i+1}, v_{2i+2}) &= 1, & \text{for } i = k, \dots, l-1, \\ x(v_{2l}, v_{2l+1}) + x(v_{2l+1}, v_1) &= 1. \end{aligned}$$

Therefore  $x(C) = l + x(v_{2k-1}, v_{2k})$ . Thus one of inequalities (7) is violated.

Finally if  $\delta^+(V(C)) = \emptyset$ , in a similar way we obtain  $x(C) = l + x(v_{2l+1}, v_1)$ .

Thus  $G$  contains no odd cycle. From Corollary 15,  $z$  is an integer, a contradiction.  $\square$

## 5. The uncapacitated facility location polytope

Now we study the case when Eq. (3) is removed from the definition of  $PC_p(G)$ . Let  $G = (V, E)$  be a graph, and let  $G'$  be the graph obtained by adding to  $G$  a new component consisting of the nodes  $u$  and  $v$  and the arc  $(u, v)$ . For a vector  $z$  associated with  $G'$  let  $z^G$  be the restriction of  $z$  to  $G$ . Let  $\Pi(G)$  be the polytope defined by (2), (4), (5), (6) and (7). Let  $\Phi_p(G')$  be the polytope defined by

$$z^G \in \Pi(G) \tag{38}$$

$$z(V \cup \{u, v\}) = p \tag{39}$$

$$z(u, v) = 1 - z(u) \tag{40}$$

$$0 \leq z(u, v) \leq z(v) = 1. \tag{41}$$

**Lemma 19.** *If  $z^*$  is an extreme point of  $\Pi(G)$ , then  $\bar{z}$  is an extreme point of  $\Phi_q(G')$ , where*

$$q = \lceil z^*(V) \rceil + 1$$

$$\bar{z}(w) = z^*(w) \quad \text{for } w \in V,$$

$$\bar{z}(w, t) = z^*(w, t) \quad \text{for } (w, t) \in E,$$

$$\bar{z}(u) = q - z^*(V) - 1$$

$$\bar{z}(u, v) = 1 - \bar{z}(u)$$

$$\bar{z}(v) = 1.$$

**Proof.** Clearly  $\bar{z} \in \Phi_q(G')$ . Suppose that  $\bar{z} = 1/2z_1 + 1/2z_2$ , with  $z_1, z_2 \in \Phi_q(G')$ . Then  $z^* = 1/2z_1^G + 1/2z_2^G$ , and  $z_1^G = z_2^G$  since  $z^*$  is an extreme point. We have  $z_1(v) = z_2(v) = 1$ . Eq. (39) implies  $z_1(u) = z_2(u)$  and Eq. (40) implies  $z_1(u, v) = z_2(u, v)$ . Thus  $\bar{z}$  is an extreme point.  $\square$

The lemma above shows that any fractional extreme point of  $\Pi(G)$  can be completed to a fractional extreme point of  $\Phi_q(G')$ . If  $G$  is  $Y$ -free then  $G'$  is also  $Y$ -free and  $\Phi_q(G')$  is integral. This shows the following.

**Theorem 20.** *If  $G$  is  $Y$ -free then the polytope  $\Pi(G)$  is integral.*

**Theorem 21.** *The  $pMP$  and the  $UFLP$  are polynomially solvable for the class of  $Y$ -free graphs.*

## Acknowledgments

Part of this work was done while the first author was visiting the T. J. Watson Research Center of IBM and while the second author was visiting the Laboratoire d'Econométrie de L'Ecole Polytechnique in Paris. The financial support provided by both institutions is greatly appreciated. We are also grateful to an anonymous referee for several helpful comments.

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